Plasma fluids from the kinetic equation

Contents

- Boltzmann equation for plasmas
- Density equation
- Momentum equation

Boltzmann equation $\quad f(t, \underset{\sim}{x}, \underset{\sim}{v})$

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \nabla f + \underbrace{\frac{\underset{\sim}{F}}{m} \cdot \frac{\partial f}{\partial \underset{\sim}{v}}}_{\frac{F_x}{m} \cdot \frac{\partial f}{\partial v_x} + \frac{F_y}{m} \cdot \frac{\partial f}{\partial v_y} + \frac{F_z}{m} \frac{\partial f}{\partial v_z}} = C(f)$$

collisions

$$\underset{\sim}{F} = q(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B})$$

$$\underset{\sim}{E}(t, \underset{\sim}{x}), \ \underset{\sim}{B}(t, \underset{\sim}{x})$$

Density $\quad$ Zeroth moment

$$n(t, \underset{\sim}{x}) = \int f(t, \underset{\sim}{x}, \underset{\sim}{v}) \, d\underset{\sim}{v}$$

number density

$$\underbrace{\int \frac{\partial f}{\partial t} d\underset{\sim}{v}}_{\frac{\partial n}{\partial t}} + \underbrace{\int \underset{\sim}{v} \cdot \nabla f \, d\underset{\sim}{v}}_{\nabla \cdot \underbrace{\int \underset{\sim}{v} f \, d\underset{\sim}{v}}_{\underset{\sim}{u} n}} + \underbrace{\int \frac{\underset{\sim}{F}}{m} \frac{\partial f}{\partial \underset{\sim}{v}} d\underset{\sim}{v}}_{\int \frac{\partial}{\partial \underset{\sim}{v}} \cdot \left(\frac{\underset{\sim}{F}}{m} f\right) d\underset{\sim}{v}} = \underbrace{\int C(f) \, d\underset{\sim}{v}}_{=0}$$

particles conserved

$$\frac{\partial F_x}{\partial v_x} = 0$$

$$\boxed{\frac{\partial n}{\partial t} + \nabla\cdot(\underset{\sim}{u}n) = 0}$$

$\rightarrow 0$ because $f \rightarrow 0$, $|\underline{v}| \rightarrow \infty$

Same as Euler equation

Momentum equation

$$\underset{\sim}{u}(t,x) = \int \underset{\sim}{v} f(t,\underset{\sim}{x},\underline{v})\, d\underline{v}$$

$$\underbrace{\int \underset{\sim}{v} \frac{\partial f}{\partial t} d\underline{v}}_{\frac{\partial}{\partial t}(\underset{\sim}{u}n)} + \underbrace{\int \underset{\sim}{v}\, \underset{\sim}{v}\cdot\nabla f\, d\underline{v}}_{\nabla\cdot\int \underset{\sim}{v}\underset{\sim}{v} f d\underline{v}} + \overset{*}{\int \underset{\sim}{v} \frac{F}{m}\cdot\frac{\partial f}{\partial \underset{\sim}{v}}} = \int \underset{\sim}{v}\, C(f)\, d\underline{v}$$

$$\nabla\cdot\int \underset{\sim}{v}\underset{\sim}{v} f d\underline{v} = \nabla\cdot\overbrace{\int \underline{\omega}\underset{\sim}{\omega} f d\omega}^{\underline{\underline{P}}/m} + \nabla\cdot(\underset{\sim}{u}\underset{\sim}{u}n)$$

$$\underline{\omega} = \underset{\sim}{v} - \underset{\sim}{u}$$

* x component

$f \rightarrow 0$ $v_y \rightarrow \pm\infty$

$$\int v_x \left(\frac{F_x}{m}\frac{\partial f}{\partial v_x} + \cancelto{0}{\frac{F_y}{m}\frac{\partial f}{\partial v_y}} + \cancelto{0}{\frac{F_z}{m}\frac{\partial f}{\partial v_z}}\right) d\underline{v}$$

$$\int v_x \frac{F_x}{m}\frac{\partial f}{\partial v_x} d\underline{v}$$

$$F_x = qE_x + q(\underset{\sim}{v}\times\underset{\sim}{B})_x$$
$$= qE_x + q(v_y B_z - v_z B_y)$$

$$= \frac{q}{m}E_x \underbrace{\int v_x \frac{\partial f}{\partial v_x} d\underline{v}}_{n(t,\underset{\sim}{x})} + \frac{q}{m}\int (v_y B_z - v_z B_y)\underbrace{v_x \frac{\partial f}{\partial v_x}}_{\rightarrow \int f dv_x} d\underline{v}$$

$$\int v_x \frac{\partial f}{\partial v_x} dv_x = \left[v_x f \right]_{-\infty}^{\infty} - \int f \, dv_x$$

$$= -\frac{q}{m} E_x n \mp \iint (v_y B_z - v_z B_y) \int f \, dv_x \, dv_y \, dv_z$$

$$= -\frac{q}{m} E_x n \mp \underbrace{\int v_y f \, d\underline{v}}_{n u_y} B_z + \underbrace{\int v_z f \, d\underline{v}}_{n u_z} B_y$$

$$= -\frac{q}{m} n \left(\underline{E} + \underline{u} \times \underline{B} \right)_x$$

Collisions

$$\frac{\partial}{\partial t}(n\underline{u}) + \nabla \cdot (\underbrace{n\underline{u}\underline{u}}_{\substack{\text{2nd moment} \\ \to \text{Energy}}}) + \nabla \cdot \underline{\underline{P}} = \frac{q}{m} n (\underline{E} + \underline{v} \times \underline{B}) + \underline{R}$$

Equations for each particle species separately

→ "Two fluid" equations